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Tensor reduction of two-loop vacuum diagrams and projectors for expanding three-point functions

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Abstract

Explicit general formulae for the tensor reduction of two-loop massive vacuum diagrams are presented. The problem of calculating the corresponding coefficients is shown to be equivalent to the problem of constructing differential operators (projectors) extracting the coefficients of the momentum expansion of massive scalar three-point functions (with any number of loops), so the solution to the latter problem is also given.

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1. Introduction

Tensor reduction of Feynman integrals containing loop momenta with uncontracted Lorentz indices in the numerator is very important for various realistic calculations in the Standard Model (and beyond). For one-loop diagrams with different numbers of external lines, several approaches and algorithms were developed [1, 2]. For two-loop self-energy diagrams, the problem was considered e.g. in ref. [3], whilst the three-point two-loop case is more complicated and requires results for the integrals with irreducible numerators [4].

The problem of finding general algorithms for the calculation of two-loop vacuum diagrams is interesting because these diagrams occur in many important physical problems. We can mention, for example, calculation of the two-loop effective potential in the Standard Model [5] and two-loop contributions to the ρ -parameter [6, 7]. Furthermore, consideration of many other problems requires constructing efficient algorithms for the momentum expansion of two-loop massive diagrams with two [8], three [9], or more external legs. The coefficients of such expansions involve massive vacuum diagrams with tensor numerators. A procedure for reducing the tensors to scalar numerators enables us to calculate these coefficients analytically, since general results (including the relevant terms of the ε -expansion) are known for the corresponding scalar diagrams [5, 6, 8, 10]. In the two-point case, the reduction is relatively simple because, as all indices are ultimately contracted with the same external momentum, only completely symmetric tensors need to be considered. However, starting from the three point-case, when there are two or more independent external momenta to contract with, a complete tensor decomposition is required.

The answer is usually easy to obtain when one has only a small number of integration momenta in the numerator. The situation becomes more complicated when one needs to calculate the higher-order coefficients of the expansion. This is the case when we are approaching the physical singularity (threshold), where the expansion still works but the parameter we expand in is already not so small. Moreover, in the approach [9] (which involves conformal mapping and Padé approximations), in order to get accurate numerical results beyond the threshold(s), one needs to calculate expansion coefficients of very high order, and therefore tensor integrals of very high rank. These are some of the reasons why general algorithms for the tensor decomposition of two-loop vacuum diagrams are important.

We would like to mention recent progress in this direction. In ref. [11], a closed expression, in terms of gamma functions, was given for arbitrary tensor vacuum integrals in which two of the masses are equal and the third is zero. Furthermore, it was noted that the algorithm used in that paper for reducing the tensors to scalars can be extended to more general cases. Some relevant formulae for diagrams with arbitrary masses, where the numerator is contracted with one or two external vectors, can be found in refs. [12, 13, 14]. Uncontracted tensor integrals were considered in refs. [14, 15]. In [14], a general expression for the matrix inverse to one containing the coefficients required for the tensor decomposition was constructed, while in [15] a version of the direct formula was presented. While solving the problem in principle, these results [14, 15] could not completely satisfy us because they were not explicitly symmetric and the coefficients required for the decomposition were rather cumbersome, so that their use in computer programs was less efficient

than recursive calculation of the corresponding coefficients. In this paper, we present a simpler general solution to this problem.

Our results also provide a complete and explicit answer to a different question, which is related to an alternative approach [9] to the expansion of scalar three-point diagrams. Instead of Taylor-expanding all propagators, one applies certain differential operators to the diagram and then sets the external momenta to zero, in order to obtain the Taylor coefficients of the diagram as a whole. In this way, one does not need to evaluate any tensor integrals, but is faced with the problem of finding the correct differential operators to use. We shall show that this problem is, in fact, equivalent to the tensor decomposition of two-loop vacuum integrals, which enables us to give a closed expression for the operator that projects out the Taylor coefficient of any given order in the three external kinematical invariants.

The remainder of this paper is organized as follows. In section 2 the notation is introduced and the connection between two-loop tensor reduction and projectors for three-point functions is established. The solutions to both problems involve a set of universal coefficients, which are calculated in section 3. A summary and a discussion of the results are contained in section 4.

2. Definitions

In this paper, we shall use the following notation:

$$I [\text{something}] \equiv \int \int d^n p \, d^n q \, \{\text{something}\} \, F(p^2, q^2, (pq)), \quad (1)$$

and we are interested in expressing the tensor integrals

$$I [p_{\mu_1} \dots p_{\mu_{N_1}} q_{\sigma_1} \dots q_{\sigma_{N_2}}] \quad (2)$$

in terms of scalar integrals. In eq. (1), n is the space-time dimension [16] and $F(p^2, q^2, (pq))$ is an arbitrary scalar function depending on Lorentz invariants of the loop momenta p and q . Usually, it is a product of (powers of) propagators,

$$(p^2 - m_1^2)^{-\nu_1} (q^2 - m_2^2)^{-\nu_2} ((p-q)^2 - m_3^2)^{-\nu_3}, \quad (3)$$

times a polynomial in p^2 , q^2 and (pq) , but the concrete form of this function will not be important for our discussion.

Because F is an even function of p and q , it is clear that the integral (2) vanishes when $N_1 + N_2$ is odd. When $N_1 + N_2$ is even (from now on, we shall always assume this), since we have no external momenta, the result must be a Lorentz invariant tensor made out of metric tensors, and it must be symmetric in the two subsets of indices, $\{\mu_1, \dots, \mu_{N_1}\}$ and $\{\sigma_1, \dots, \sigma_{N_2}\}$. A basis of tensor structures with these properties can be described as follows.

Each structure is characterized by three integers, j_1 , j_2 and j_3 , such that $2j_1 + j_3 = N_1$ and $2j_2 + j_3 = N_2$. It can be constructed by taking a product of j_1 metric tensors $g_{\mu_i \mu_k}$, j_2 tensors $g_{\sigma_i \sigma_k}$ and j_3 tensors $g_{\mu_i \sigma_k}$, and then symmetrizing it in the μ 's and σ 's by taking the sum of all *distinct* products of metric tensors we get from it through permutations of

the μ 's and of the σ 's:

$$\begin{aligned} & \{j_1, j_2, j_3\}_{\mu_1 \dots \mu_{N_1} \sigma_1 \dots \sigma_{N_2}} \\ & \equiv g_{\mu_1 \mu_2} \dots g_{\mu_{2j_1-1} \mu_{2j_1}} g_{\sigma_1 \sigma_2} \dots g_{\sigma_{2j_2-1} \sigma_{2j_2}} g_{\mu_{2j_1+1} \sigma_{2j_2+1}} \dots g_{\mu_{2j_1+j_3} \sigma_{2j_2+j_3}} \\ & + \text{permutations} . \end{aligned} \quad (4)$$

The number of terms on the r.h.s. is

$$t_{j_1 j_2 j_3} = \frac{N_1! N_2!}{2^{j_1+j_2} j_1! j_2! j_3!} . \quad (5)$$

Equivalent representations of the tensor structures (4) are

$$\begin{aligned} \{j_1, j_2, j_3\} &= \frac{t_{j_1 j_2 j_3}}{N_1! N_2!} \frac{\partial^{N_1}}{\partial k_1^{\mu_1} \dots \partial k_1^{\mu_{N_1}}} \frac{\partial^{N_2}}{\partial k_2^{\sigma_1} \dots \partial k_2^{\sigma_{N_2}}} (k_1^2)^{j_1} (k_2^2)^{j_2} (k_1 k_2)^{j_3} \\ &= \frac{t_{j_1 j_2 j_3}}{N_1! N_2!} \square_{11}^{j_1} \square_{22}^{j_2} \square_{12}^{j_3} k_{1\mu_1} \dots k_{1\mu_{N_1}} k_{2\sigma_1} \dots k_{2\sigma_{N_2}} , \end{aligned} \quad (6)$$

where we have adopted the notation of [9] for d'Alembertian operators,

$$\square_{ij} = g^{\mu\nu} \frac{\partial^2}{\partial k_i^\mu \partial k_j^\nu} . \quad (7)$$

For simplicity, we have suppressed the Lorentz indices on the l.h.s. of (6). We note that in [2, 11]³, d'Alembertians were used in a similar way to generate the required tensor structures.

At given N_1 and N_2 , the number of independent tensor structures (4) is

$$T(N_1, N_2) = 1 + \min([N_1/2], [N_2/2]) , \quad (8)$$

where $[N_i/2]$ is the integer part of $N_i/2$, and each of these structures is already determined by one of the j 's, e.g., j_3 . Because we shall often need to sum over the set of all tensor structures, we introduce the following notation:

$$\sum_{\{j\}} \equiv \sum_{\substack{j_1, j_2, j_3 \\ 2j_1+j_3=N_1 \\ 2j_2+j_3=N_2}} , \quad (9)$$

emphasizing that this is actually just a *single* sum.

Now, we return to the integral (2) and write it as a linear combination

$$I \left[p_{\mu_1} \dots p_{\mu_{N_1}} q_{\sigma_1} \dots q_{\sigma_{N_2}} \right] = \sum_{\{j\}} \{j_1, j_2, j_3\} I_{j_1 j_2 j_3} , \quad (10)$$

with some scalar coefficients $I_{j_1 j_2 j_3}$ which are to be determined. By contracting (10) with each of the structures (4) we obtain the following system of $T(N_1, N_2)$ linear equations for the $I_{j_1 j_2 j_3}$'s:

$$t_{j_1 j_2 j_3} I \left[(p^2)^{j_1} (q^2)^{j_2} (pq)^{j_3} \right] = \sum_{\{j'\}} \chi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} I_{j'_1 j'_2 j'_3} , \quad (11)$$

³ Here, we would like to mention that eq. (B.4) of [17] is equivalent to the result presented in [2], if one explicitly calculates the result of applying the d'Alembertians.

where

$$\chi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} = \{j_1, j_2, j_3\}_{\mu_1 \dots \mu_{N_1} \sigma_1 \dots \sigma_{N_2}} \{j'_1, j'_2, j'_3\}^{\mu_1 \dots \mu_{N_1} \sigma_1 \dots \sigma_{N_2}} . \quad (12)$$

It is useful to think of the coefficients $\chi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ as the elements of a $T(N_1, N_2) \times T(N_1, N_2)$ “contraction matrix”, whose rows and columns are labelled by the triplets (j_1, j_2, j_3) and (j'_1, j'_2, j'_3) , respectively. We shall denote the inverse of this matrix by ϕ , i.e.

$$\sum_{\{j''\}} \chi_{j_1 j_2 j_3; j''_1 j''_2 j''_3} \phi_{j''_1 j''_2 j''_3; j'_1 j'_2 j'_3} = \delta_{j_1 j_2 j_3; j'_1 j'_2 j'_3} \equiv \delta_{j_1 j'_1} \delta_{j_2 j'_2} \delta_{j_3 j'_3} . \quad (13)$$

Since χ is a symmetric matrix, ϕ is also symmetric:

$$\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} = \phi_{j'_1 j'_2 j'_3; j_1 j_2 j_3} . \quad (14)$$

Using ϕ to solve the system of equations (11), we can re-write eq. (10) as

$$I \left[p_{\mu_1} \dots p_{\mu_{N_1}} q_{\sigma_1} \dots q_{\sigma_{N_2}} \right] = \sum_{\{j\}} \{j_1, j_2, j_3\} \sum_{\{j'\}} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} t_{j'_1 j'_2 j'_3} I \left[(p^2)^{j'_1} (q^2)^{j'_2} (pq)^{j'_3} \right] . \quad (15)$$

Thus, we have reduced the tensor integrals (2) to a combination of scalar integrals (carrying the same total powers of momenta p and q) multiplied by the tensor structures $\{j_1, j_2, j_3\}$. In the next section, we shall derive explicit expressions for both ϕ and χ .

Here, we address the second question posed in the introduction. Consider a massive, scalar, three-point Feynman diagram depending on two independent external momenta k_1 and k_2 , and suppose we are interested in the coefficients of its Taylor expansion,

$$C(k_1, k_2) = \sum_{j_1, j_2, j_3} c_{j_1 j_2 j_3} (k_1^2)^{j_1} (k_2^2)^{j_2} (k_1 k_2)^{j_3} . \quad (16)$$

Our goal is to construct a set of projection operators that give us the coefficients $c_{j_1 j_2 j_3}$, using derivatives with respect to the *vectors* k_1 and k_2 . This will allow us to apply them to the integrand of the Feynman integral through which $C(k_1, k_2)$ is defined.

Following [9], let us apply a product of d'Alembertians to both sides of (16), and then set $k_1 = k_2 = 0$. This gives

$$\begin{aligned} \square_{11}^{j_1} \square_{22}^{j_2} \square_{12}^{j_3} C(k_1, k_2) \Big|_{k_1=k_2=0} &= \sum_{\{j'\}} c_{j'_1 j'_2 j'_3} \square_{11}^{j'_1} \square_{22}^{j'_2} \square_{12}^{j'_3} (k_1^2)^{j'_1} (k_2^2)^{j'_2} (k_1 k_2)^{j'_3} \\ &= N_1! N_2! \sum_{\{j'\}} \frac{\chi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}}{t_{j_1 j_2 j_3} t_{j'_1 j'_2 j'_3}} c_{j'_1 j'_2 j'_3} , \end{aligned} \quad (17)$$

where $N_1 = 2j_1 + j_3$ and $N_2 = 2j_2 + j_3$. The second line follows from eqs. (6) and (12). Solving the system of equations (17) gives us the operators we are looking for:

$$c_{j_1 j_2 j_3} = \frac{t_{j_1 j_2 j_3}}{N_1! N_2!} \sum_{\{j'\}} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} t_{j'_1 j'_2 j'_3} \square_{11}^{j'_1} \square_{22}^{j'_2} \square_{12}^{j'_3} C(k_1, k_2) \Big|_{k_1=k_2=0} , \quad (18)$$

where the coefficients $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ are the same as the ones needed in the tensor reduction formula (15)! For the coefficients c_{00j_3} and $c_{j_1 0 j_3}$, this result, combined with eq. (27) presented below, coincides with the expressions given in [9].

Another way to see the connection between (15) and (18) is by taking the scalar function $C(k_1, k_2)$ in the latter to be $I \left[(k_1 p)^{N_1} (k_2 q)^{N_2} \right]$, and noting that

$$I \left[p_{\mu_1} \dots p_{\mu_{N_1}} q_{\sigma_1} \dots q_{\sigma_{N_2}} \right] = \frac{1}{N_1! N_2!} \frac{\partial^{N_1}}{\partial k_1^{\mu_1} \dots \partial k_1^{\mu_{N_1}}} \frac{\partial^{N_2}}{\partial k_2^{\sigma_1} \dots \partial k_2^{\sigma_{N_2}}} I \left[(k_1 p)^{N_1} (k_2 q)^{N_2} \right]. \quad (19)$$

From this point of view, it is also clear that the connection can immediately be generalized: the ϕ 's occurring in the tensor reduction of L -loop vacuum diagrams are the same as in the projectors for the Taylor coefficients of $(L+1)$ -point scalar diagrams.

3. Results

We calculate the coefficients $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ by an inductive method, based on recurrence relations, which enables us to express them in a compact and symmetric way. Afterwards, we shall obtain χ by inverting ϕ .

We start from the following simple formulae for contracting two of the Lorentz indices of the tensor structures (4), which can be derived, e.g., by using (6):

$$g_{\mu_{N_1} \sigma_{N_2}} \{j_1, j_2, j_3\} = (n+2j_1+2j_2+j_3-1) \{j_1, j_2, j_3-1\} + (j_3+1) \{j_1-1, j_2-1, j_3+1\}, \quad (20)$$

$$g_{\mu_{N_1-1} \mu_{N_1}} \{j_1, j_2, j_3\} = (n+2j_1+2j_3-2) \{j_1-1, j_2, j_3\} + 2(j_2+1) \{j_1, j_2+1, j_3-2\} \quad (21)$$

and an analogous formula for contraction with $g_{\sigma_{N_2-1} \sigma_{N_2}}$. On the r.h.s., the structures $\{j_1, j_2, j_3\}$ should be set to zero whenever any of the j 's becomes negative. Using these contractions, the following recurrence relations for $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ can be obtained:

$$\begin{aligned} N_1 N_2 \left\{ (n+N_1+N_2-j'_3-1) \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} + (j'_3-1) \phi_{j_1 j_2 j_3; j'_1+1, j'_2+1, j'_3-2} \right\} \\ = j_3 \phi_{j_1, j_2, j_3-1; j'_1, j'_2, j'_3-1} \quad (j'_3 > 0), \quad (22) \end{aligned}$$

$$\begin{aligned} N_1 (N_1-1) \left\{ (n+N_1+j'_3-2) \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} + 2j'_2 \phi_{j_1 j_2 j_3; j'_1-1, j'_2-1, j'_3+2} \right\} \\ = 2 j_1 \phi_{j_1-1, j_2, j_3; j'_1-1, j'_2, j'_3} \quad (j'_1 > 0), \quad (23) \end{aligned}$$

and also a relation similar to (23) but with interchanged indices, $1 \leftrightarrow 2$. If either the second ϕ on the l.h.s. or the one on the r.h.s. has a negative index, it is to be set to zero.

Let us consider some fixed values for j_1 , j_2 and j_3 and suppose we already know the ϕ 's on the r.h.s. of, say, (22), corresponding to (N_1-1) μ 's and (N_2-1) σ 's. The number of equations (22) is $T(N_1-1, N_2-1)$. If the number of tensor structures does not increase between (N_1-1, N_2-1) and (N_1, N_2) (i.e., when N_1 and N_2 are odd), this means we have enough relations to determine the ϕ 's on the l.h.s. completely. On the other hand, if at this step one extra tensor appears, we need one more relation between the ϕ 's. The same is true if we want to use (23) to go from (N_1-2, N_2) to (N_1, N_2) . The additional information can be obtained by contracting (15) with $k^{\mu_1} \dots k^{\mu_{N_1}} k^{\sigma_1} \dots k^{\sigma_{N_2}}$ and using (cf. eq. (B.10) of [12])

$$I \left[(kp)^{N_1} (kq)^{N_2} \right] = \frac{(k^2)^{(N_1+N_2)/2}}{2^{(N_1+N_2)/2} \binom{n}{2}_{(N_1+N_2)/2}} \sum_{\{j\}} t_{j_1 j_2 j_3} I \left[(p^2)^{j_1} (q^2)^{j_2} (pq)^{j_3} \right], \quad (24)$$

where $(a)_j \equiv \Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol. Comparing the coefficients of $I[(p^2)^{j_1}(q^2)^{j_2}(pq)^{j_3}]$, we see that

$$\sum_{\{j'\}} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} t_{j'_1 j'_2 j'_3} = \frac{1}{2^{(N_1+N_2)/2} \left(\frac{n}{2}\right)_{(N_1+N_2)/2}}. \quad (25)$$

The recurrence relations (22)–(23) can be solved in several ways, leading to different representations for ϕ . First, we consider the special case when $j_1 = 0$, so that the r.h.s. of eq. (23) vanishes and we get a relation between $\phi_{0j_2j_3;j'_1j'_2j'_3}$ and $\phi_{0j_2j_3;j'_1-1,j'_2-1,j'_3+2}$. Repeated application of this relation yields

$$\phi_{0j_2j_3;j'_1j'_2j'_3} = \frac{(-1)^{j'_1} j'_2!}{j_2! \left(\frac{n-2}{2} + j'_1 + j'_3\right)_{j'_1}} \phi_{0j_2j_3;0j_2j_3}. \quad (26)$$

To determine the normalization, we insert this result in (25). The sum is a terminating ${}_2F_1$ series of unit argument, which can be written in terms of Pochhammer symbols. Substituting the expression for $\phi_{0j_2j_3;0j_2j_3}$ thus found back into (26) and using the symmetry (14) gives

$$\phi_{j_1 j_2 j_3; 0, (N_2-N_1)/2, N_1} = \frac{(-1)^{j_1} j_2! \left(\frac{n-2}{2}\right)_{j_1+j_3}}{N_2! \left(\frac{n}{2}\right)_{(N_1+N_2)/2} (n-2)_{N_1}}. \quad (27)$$

To handle the general case, we move the second term in (23) to the r.h.s. In this way, we get a recurrence relation in j'_1 , which has the following solution:

$$\begin{aligned} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} &= \frac{j_1! j'_1! j'_2!}{N_1!} \sum_{l=\max(0, j'_1-j_1)}^{\min(j'_1, j'_2)} \frac{(-1)^l (j'_3 + 2l)!}{l! (j'_1 - l)! (j'_2 - l)! (j_1 - j'_1 + l)!} \\ &\times \frac{\left(\frac{n}{2}\right)_{j'_3+2l} \left(\frac{n-2}{2}\right)_{j'_3+l}}{\left(\frac{n-2}{2}\right)_{j'_3+2l} \left(\frac{n}{2}\right)_{j'_1+j'_3+l}} \phi_{j_1-j'_1+l, j_2, j_3; 0, j'_2-l, j'_3+2l}. \end{aligned} \quad (28)$$

We can write this as a familiar sum over j 's, (9), by defining $j''_1 = j'_1 - l$, $j''_2 = j'_2 - l$ and $j''_3 = j'_3 + 2l$. Substituting the result for $j'_1 = 0$ (27), we find

$$\begin{aligned} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} &= \frac{j_1! j_2! j'_1! j'_2! (-1)^{(j_3-j'_3)/2}}{N_1! N_2!} \sum_{\{j''\}} \frac{j''_3!}{j''_1! j''_2! \left(\frac{j''_3-j'_3}{2}\right)! \left(\frac{j''_3-j'_3}{2}\right)!} \\ &\times \frac{\left(\frac{n}{2}\right)_{j''_3} \left(\frac{n-2}{2}\right)_{(j_3+j''_3)/2} \left(\frac{n-2}{2}\right)_{(j'_3+j''_3)/2}}{\left(\frac{n-2}{2}\right)_{j''_3} (n-2)_{j''_3} \left(\frac{n}{2}\right)_{j''_1+j''_3} \left(\frac{n}{2}\right)_{j''_2+j''_3}}. \end{aligned} \quad (29)$$

In (29) and in the rest of this section, it is understood that terms with factorials of negative integers in the denominator vanish. Note that (29) clearly shows the symmetries $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} = \phi_{j'_1 j'_2 j'_3; j_1 j_2 j_3} = \phi_{j_2 j_1 j_3; j'_2 j'_1 j'_3}$.

Another special case for which we find a simple expression is when $j_3 = 0$. This time, the r.h.s. of (22) vanishes, and steps similar to the ones described above lead us to

$$\phi_{j_1 j_2 j_3; N_1/2, N_2/2, 0} = \frac{\left(\frac{N_1}{2}\right)! \left(\frac{N_2}{2}\right)! (-1)^{j_3/2} j_3! \left(\frac{n-1}{2}\right)_{(N_1+N_2-j_3)/2}}{N_1! N_2! 2^{j_3} \left(\frac{j_3}{2}\right)! \left(\frac{n-1}{2}\right)_{N_1/2} \left(\frac{n-1}{2}\right)_{N_2/2} \left(\frac{n}{2}\right)_{(N_1+N_2)/2}}. \quad (30)$$

Using (22) as a recurrence relation in j'_3 gives the following result:

$$\begin{aligned} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} &= \frac{j_3! j'_3!}{N_1! N_2!} \sum_{l=\max(0, j_1-j'_1)}^{[j'_3/2]} \frac{(-1)^l (2j'_1 + 2l)! (2j'_2 + 2l)!}{4^l l! (j'_3 - 2l)! (j_3 - j'_3 + 2l)!} \\ &\times \frac{\left(\frac{n-1}{2}\right)_{j'_1+j'_2+l} (n)_{2j'_1+2j'_2+4l}}{\left(\frac{n-1}{2}\right)_{j'_1+j'_2+2l} (n)_{2j'_1+2j'_2+j'_3+2l}} \phi_{j_1, j_2, j_3-j'_3+2l; j'_1+l, j'_2+l, 0}. \end{aligned} \quad (31)$$

By writing $j''_1 = j'_1 + l$, $j''_2 = j'_2 + l$ and $j''_3 = j'_3 - 2l$, and inserting (30) we obtain a second representation for ϕ :

$$\begin{aligned} \phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} &= \frac{j_3! j'_3! (-1)^{(j_3-j'_3)/2} 2^{j_1+j_2+j'_1+j'_2}}{N_1! N_2!} \sum_{\{j''\}} \frac{j''_1! j''_2!}{j''_3! \left(\frac{j_3-j''_3}{2}\right)! \left(\frac{j'_3-j''_3}{2}\right)!} \\ &\times \frac{\left(\frac{n+1}{2}\right)_{j''_1+j''_2} \left(\frac{n-1}{2}\right)_{(j_1+j_2+j''_1+j''_2)/2} \left(\frac{n-1}{2}\right)_{(j'_1+j'_2+j''_1+j''_2)/2}}{\left(\frac{n-1}{2}\right)_{j''_1+j''_2} \left(\frac{n-1}{2}\right)_{j''_1} \left(\frac{n-1}{2}\right)_{j''_2} (n)_{2j''_1+2j''_2+j''_3}}. \end{aligned} \quad (32)$$

The representations (29) and (32) complement each other. The number of non-vanishing terms in (29) is $1 + \min(j_1, j_2, j'_1, j'_2)$, whereas the number in (32) is $1 + \min([j_3/2], [j'_3/2])$. Therefore, depending on the values of the j 's and j' 's, it can be much more efficient to use one or the other, even though both lead to identical results⁴. A simple example is when N_1 and N_2 are odd and $j'_3 = 1$. In this case, (32) consists of just one term, with $j''_3 = 1$, which gives

$$\phi_{j_1 j_2 j_3; (N_1-1)/2, (N_2-1)/2, 1} = \frac{\left(\frac{N_1-1}{2}\right)! \left(\frac{N_2-1}{2}\right)! (-1)^{(j_3-1)/2} j_3! \left(\frac{n-1}{2}\right)_{(N_1+N_2-j_3-1)/2}}{N_1! N_2! 2^{j_3} \left(\frac{j_3-1}{2}\right)! \left(\frac{n-1}{2}\right)_{(N_1-1)/2} \left(\frac{n-1}{2}\right)_{(N_2-1)/2} \left(\frac{n}{2}\right)_{(N_1+N_2)/2}}. \quad (33)$$

An interesting feature of (29) and (32) is that, in each term of the sum, the dependence on j and on j' is factorized. Both formulae have the structure $\phi = M^T D M$, where M is a triangular matrix and D is diagonal. The matrix elements of the matrices M , and of their inverses M^{-1} , can be presented as

$$M_{jl} = \frac{(a)_{j+l}}{(l-j)!} b_l, \quad (M^{-1})_{jl} = \frac{(-1)^{j-l} (a+1)_{2j}}{(l-j)! (a)_{2j} (a+1)_{j+l}} \frac{1}{b_j}. \quad (34)$$

⁴ It is possible to verify that (29) reduces to (30) when $j'_3 = 0$, and (32) to (27) when $j'_1 = 0$, with the help of a theorem on the summation of ${}_5F_4$ series; see [18], §4.3, eq. 3, or [19], app. III, eq. 13.

With this decomposition, it is now easy to invert the matrix ϕ , which gives us the following representations for the contraction matrix χ :

$$\chi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} = \frac{N_1! N_2!}{j_1! j_2! j'_1! j'_2!} \sum_{\{j''\}} \frac{j_1''! j_2''!}{j_3''! \left(\frac{j_3-j_3''}{2}\right)! \left(\frac{j'_3-j_3''}{2}\right)!} \frac{(n-2)_{j_3''} \left(\frac{n}{2}\right)_{j_3''} \left(\frac{n}{2}\right)_{j_1''+j_3''} \left(\frac{n}{2}\right)_{j_2''+j_3''}}{\left(\frac{n-2}{2}\right)_{j_3''} \left(\frac{n}{2}\right)_{(j_3+j_3'')/2} \left(\frac{n}{2}\right)_{(j'_3+j_3'')/2}} \quad (35)$$

if we use (29), and

$$\begin{aligned} \chi_{j_1 j_2 j_3; j'_1 j'_2 j'_3} &= \frac{N_1! N_2!}{2^{j_1+j_2+j'_1+j'_2} j_3! j'_3!} \sum_{\{j''\}} \frac{j_3''!}{j_1''! j_2''! \left(\frac{j_3-j_3''}{2}\right)! \left(\frac{j'_3-j_3''}{2}\right)!} \\ &\times \frac{(n)_{2j_1''+2j_2''+j_3''} \left(\frac{n-1}{2}\right)_{j_1''} \left(\frac{n-1}{2}\right)_{j_2''} \left(\frac{n+1}{2}\right)_{j_1''+j_2''}}{\left(\frac{n-1}{2}\right)_{j_1''+j_2''} \left(\frac{n+1}{2}\right)_{(j_1+j_2+j_1''+j_2'')/2} \left(\frac{n+1}{2}\right)_{(j'_1+j'_2+j_1''+j_2'')/2}} \end{aligned} \quad (36)$$

if we use (32). We also note that the determinant of ϕ is given by the product of the diagonal elements of M^T , D and M ,

$$\det \phi = \prod_{\{j\}} \frac{j_1! j_2! j_3! \left(\frac{n-2}{2}\right)_{j_3} \left(\frac{n}{2}\right)_{j_3}}{N_1! N_2! (n-2)_{j_3} \left(\frac{n}{2}\right)_{j_1+j_3} \left(\frac{n}{2}\right)_{j_2+j_3}}. \quad (37)$$

The matrices M have a rather nice interpretation. They transform the basis of tensor structures $\{j_1, j_2, j_3\}$ into orthogonal bases. The orthogonal tensor structures involved in our first representation, (29) and (35), can be written as

$$\begin{aligned} \{\{N_1, N_2; j_3\}\} &\equiv 2^{(N_1+N_2)/2} \sum_{\{j'\}} (-1)^{(j'_3-j_3)/2} \frac{j'_1! j'_2!}{\left(\frac{j_3-j'_3}{2}\right)!} \left(\frac{n-2}{2}\right)_{(j_3+j'_3)/2} \{j'_1, j'_2, j'_3\} \\ &= \frac{\partial^{N_1}}{\partial k_1^{\mu_1} \dots \partial k_1^{\mu_{N_1}}} \frac{\partial^{N_2}}{\partial k_2^{\sigma_1} \dots \partial k_2^{\sigma_{N_2}}} (k_1^2)^{N_1/2} (k_2^2)^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{(k_1 k_2)}{\sqrt{k_1^2 k_2^2}} \right) \\ &= \square_{11}^{N_1/2} \square_{22}^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{\square_{12}}{\sqrt{\square_{11} \square_{22}}} \right) k_{1\mu_1} \dots k_{1\mu_{N_1}} k_{2\sigma_1} \dots k_{2\sigma_{N_2}}, \end{aligned} \quad (38)$$

where

$$C_j^\gamma(x) = \sum_{l=0}^{[j/2]} \frac{(-1)^l (\gamma)_{j-l}}{l!(j-2l)!} (2x)^{j-2l} \quad (39)$$

are Gegenbauer polynomials. If we remember that $N_1 = 2j_1 + j_3$ and $N_2 = 2j_2 + j_3$, we see that (38) does not really contain negative powers or square roots of d'Alembertians. The orthogonality of the structures (38) can be explained using the fact [20] that

$$\square_{11} (k_1^2)^{j_3/2} (k_2^2)^{j_3/2} C_{j_3}^{(n-2)/2} \left(\frac{(k_1 k_2)}{\sqrt{k_1^2 k_2^2}} \right) = 0 \quad (40)$$

(and the same for \square_{22}), and it is closely related to the orthogonality of the Gegenbauer polynomials with respect to integration over the unit sphere in n -dimensional Euclidean space.

An analogous explanation can be given for our second representation, (32) and (36). The corresponding orthogonal tensor structures are connected with polynomials that are annihilated by \square_{12} ,

$$(k_1^2)^{j_1} (k_2^2)^{j_2} \sum_{l=0}^{\min(j_1, j_2)} \frac{(-1)^l \left(\frac{n-1}{2}\right)_{j_1+j_2-l}}{l! (j_1-l)! (j_2-l)!} \left(\frac{(k_1 k_2)^2}{k_1^2 k_2^2}\right)^l. \quad (41)$$

For completeness, we mention that a third representation for $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ can be derived by using (23) in the opposite direction, as a recurrence relation in j'_3 . It is a sum of $1 + \min(j_1, [j'_3/2])$ terms, in which the dependence on j and j' is factorized. However, it is less symmetric than (29) and (32) and does not appear to have such a simple interpretation. It is also possible to write (22) as a recurrence relation in j'_1 , but it is more difficult to solve than the other ones.

To conclude this section, let us insert our first expression for ϕ (29) into the reduction formula (15). The result can be written in the following way⁵:

$$\begin{aligned} I \left[p_{\mu_1} \cdots p_{\mu_{N_1}} q_{\sigma_1} \cdots q_{\sigma_{N_2}} \right] &= \frac{1}{2^{N_1+N_2}} \sum_{\{j\}} \frac{j_3!}{j_1! j_2!} \frac{\left(\frac{n}{2}\right)_{j_3}}{\left(\frac{n-2}{2}\right)_{j_3} (n-2)_{j_3} \left(\frac{n}{2}\right)_{j_1+j_3} \left(\frac{n}{2}\right)_{j_2+j_3}} \\ &\times I \left[(p^2)^{N_1/2} (q^2)^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{(pq)}{\sqrt{p^2 q^2}} \right) \right] \{ \{N_1, N_2; j_3\} \}. \end{aligned} \quad (42)$$

When contracted with k_1 and k_2 , this gives

$$\begin{aligned} I \left[(k_1 p)^{N_1} (k_2 q)^{N_2} \right] &= \frac{N_1! N_2!}{2^{N_1+N_2}} \sum_{\{j\}} \frac{j_3!}{j_1! j_2!} \frac{\left(\frac{n}{2}\right)_{j_3}}{\left(\frac{n-2}{2}\right)_{j_3} (n-2)_{j_3} \left(\frac{n}{2}\right)_{j_1+j_3} \left(\frac{n}{2}\right)_{j_2+j_3}} \\ &\times I \left[(p^2)^{N_1/2} (q^2)^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{(pq)}{\sqrt{p^2 q^2}} \right) \right] (k_1^2)^{N_1/2} (k_2^2)^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{(k_1 k_2)}{\sqrt{k_1^2 k_2^2}} \right), \end{aligned} \quad (43)$$

and, similarly, the three-point functions (16) can be expressed as

$$\begin{aligned} C(k_1, k_2) &= \sum_{N_1, N_2} \frac{1}{2^{N_1+N_2}} \sum_{\{j\}} \frac{j_3!}{j_1! j_2!} \frac{\left(\frac{n}{2}\right)_{j_3}}{\left(\frac{n-2}{2}\right)_{j_3} (n-2)_{j_3} \left(\frac{n}{2}\right)_{j_1+j_3} \left(\frac{n}{2}\right)_{j_2+j_3}} \\ &\times (k_1^2)^{N_1/2} (k_2^2)^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{(k_1 k_2)}{\sqrt{k_1^2 k_2^2}} \right) \\ &\times \left\{ \square_{11}^{N_1/2} \square_{22}^{N_2/2} C_{j_3}^{(n-2)/2} \left(\frac{\square_{12}}{\sqrt{\square_{11} \square_{22}}} \right) C(k_1, k_2) \right\}_{k_1=k_2=0}. \end{aligned} \quad (44)$$

⁵ In eq.(6) of ref. [15], the scalar coefficients are also written in terms of Gegenbauer polynomials, but a different basis is used for the tensor structures. By some transformations, and the summation of a hypergeometric series of unit argument, one can show that the result is equivalent to (42).

4. Conclusion

The main results of this paper are the explicit formulae (29) and (32) for the coefficients $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$, which are necessary for both the reduction of two-loop vacuum tensor integrals of arbitrary rank to scalar integrals (15), and for the construction of the projectors (18) that extract the Taylor coefficients of scalar three-point diagrams. The tensor integrals are written in a more compact way (42) by using orthogonal bases involving Gegenbauer polynomials for the tensor structures and scalar integrals. A similar representation is also derived for the three-point functions, eq. (44). This shows, once again, that when one expands in $(k_1 k_2)$, the basis of the Gegenbauer polynomials is useful [20] (see also in [15]). On the way, we have also obtained explicit expressions for the elements of the contraction matrix χ , (35) and (36).

The elements of the decomposition matrix ϕ are expressed as single finite sums of quotients of Pochhammer symbols involving the space-time dimension n . Whenever one of the indices j or j' reaches its minimal value (i.e. $j_1 = 0$, $j_2 = 0$, $j_3 = 0$ or $j_3 = 1$), we have just a single term (eqs. (27), (30) and (33)). In most cases, however, a sum of several terms cannot be avoided because the numerator of $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ contains non-factorizable quadratic (or higher degree) factors, e.g.:

$$\phi_{1,1,2;1,1,2} = \frac{n^2 + 3n + 6}{72(n-1)n(n+1)(n+2)(n+4)(n+6)} . \quad (45)$$

Therefore, it seems unlikely that more simple general formulae for $\phi_{j_1 j_2 j_3; j'_1 j'_2 j'_3}$ than (29) and (32) can be found.

Numerous checks on the results were performed by computer algebra [21]. We also verified, for a large number of cases, that the reduction formula in ref. [15] is equivalent to ours. The reduction algorithm in [11] is of a different kind, because it is based on integration by parts [2] and depends on the particular form of the propagators in the diagrams.

Let us now briefly discuss the application of these formulae. As an example, we consider the momentum expansion of a massive two-loop three-point function (see e.g. in [9, 15]). Here, one can choose between two methods. The first is to expand all propagators in powers of the external momenta, yielding a collection of vacuum integrals with tensor numerators. Then, all the tensor integrals are reduced to scalar integrals times invariant tensor structures $\{j_1, j_2, j_3\}$. After that, the integrals can be calculated by expressing the numerators in terms of inverse propagators and cancelling them against the denominators. The final step is to contract the tensor structures with the external vectors (momenta or polarization vectors), which gives us a polynomial in scalar products of those vectors. If two or more of the external vectors are identical, each term is accompanied by a symmetry factor which, in general, can be written as a multiple sum of combinations of factorials, or computed by a recursive algorithm.

The second method is to decompose the three-point function into scalar form factors, which are then expanded in the external invariants by using d'Alembertian operators. It takes some work to apply the d'Alembertians to the integrand, but in return, we no longer need to contract any tensor structures with external vectors. After setting the external momenta to zero, we get scalar vacuum integrals, which are evaluated in the same way as before. For given N_1 and N_2 , we end up with three sums: one over powers of the

external momenta, one over powers of the d'Alembertians, and the internal sum in the representation of ϕ . As we have already mentioned, two of these sums can be absorbed in Gegenbauer polynomials, in which case only one extra sum remains.

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